

SUBGROUP MAJORIZATION

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ABSTRACT. The extension of majorization (also called the rearrangement ordering), to more general groups than the symmetric (permutation) group, is referred to as G -majorization. There are strong results in the case that G is a reflection group and this paper builds on this theory in the direction of subgroups, normal subgroups, quotient groups and extensions. The implications for fundamental cones and order-preserving functions are studied. The main example considered is the hyperoctahedral group, which, acting on a vector in \mathbb{R}^n , permutes and changes the signs of components.

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1. INTRODUCTION

Majorization is now the general term for the study of inequalities which began with the theory of rearrangements expounded at length by Hardy, Littlewood and Polya [10] and given impetus by the book of Marshall and Olkin [13], now in its second expanded edition [14]. The group of permutations, the symmetric group S_n , is at the heart of this classical majorization, and a major advance was the extension to generalised or G -majorization which applies particularly to general reflection groups (Eaton and Perlman [7]). The present paper is a contribution to G -majorization. Following a short introduction, we investigate the implication of a number of group operations, in particular the restriction to subgroups, quotients and extensions.

We begin with the basic definition.

Definition 1.1. *Let \mathcal{X} be an n -dimensional Euclidean space and let G be a finite matrix group operating on \mathcal{X} . We define a partial ordering on \mathcal{X} , written $y \prec_G x$ by*

$$y \in \text{conv}(\{gx : g \in G\})$$

Here conv is the convex hull and $\mathcal{O}_G(x) = \{gx : g \in G\}$ is the orbit of x in \mathcal{X} under the action of G . For classical majorization G is the symmetric group S_n , and the action of G permutes coordinates. That is, the action of $g \in G$ permutes the entries of x .

The following is a basic duality result for G -majorization [3, 9]. We use $\langle \cdot, \cdot \rangle$ to denote the Euclidean inner product and define

$$m(z, x) = \sup_{g \in G} \langle z, g(x) \rangle.$$

Theorem 1.2. *Let G be a closed subgroup of the orthogonal group O_n , acting on \mathcal{X} . Then $y \prec_G x$ is equivalent to*

$$m(z, y) \leq m(z, x), \text{ for all } z \in \mathcal{X}$$

It will be convenient to slightly extend the convex hull definition in Definition 1.1 as part of the discussion on G -majorization as a cone ordering, below.

1.1. Reflection groups. The main results concerning extension of majorization are for the extension from the symmetric group, the classical majorization case, to reflection groups. The essence is contained in Theorem 1.3, below, the major credit for which should go to Eaton and Perlman and Eaton [7, 2, 3, 5, 4, 6, 8]. Giovagnoli and Wynn [9] made contributions working in the context of the extension of majorization to spaces of matrices. These authors realised the importance of the fundamental cone of the reflection groups. An important question had remained as to whether the equivalent conditions of Theorem 1.3 applied *only* for reflection groups and this was answered in the affirmative by Steerneman [16] who revisited the theory with careful discussion of many equivalent conditions. Thus the machinery of G -majorization was established.

Finite reflection groups acting on Euclidean space are classified according to the finite Coxeter groups, defined by having a generating set S with relations $s^2 = e$, the identity, for all $s \in S$, and $(s_i s_j)^{m_{ij}} = e$ for $s_i, s_j \in S$ and with m_{ij} integers ≥ 2 (see [11] for instance, for more details). Any finite reflection group G also has a representation as a subgroup of the orthogonal group O_n acting on $\mathcal{X} = \mathbb{R}^n$, for n sufficiently large. We shall fix n and consider the class \mathcal{G} of all reflection groups acting in this way on \mathcal{X} .

Any $G \in \mathcal{G}$ is defined by a finite set of distinct generating hyperplanes V_j , defined by

$$V_j = \{x, \langle x, a_j \rangle = 0, j = 1, \dots, k\},$$

where the a_j are the positive roots in the root system of G . These define half spaces

$$V_j^+ = \{x, \langle x, a_j \rangle \geq 0, j = 1, \dots, k\}$$

which in turn define the fundamental cone

$$\mathcal{C}_G = \bigcap_{i=1}^k V_i^+.$$

A fundamental *region* R has the defining properties (i) R is open (ii) for any $x \in R$ there is no other $x' = g(x) \in R$ for any $g \in G$, (iii) $\mathcal{X} = \bigcup_{g \in G} g(\bar{R})$, where the bar means closure. For a finite reflection group $G \in \mathcal{G}$ the interior of its fundamental cone \mathcal{C}_G° is a fundamental region.

The fundamental cone is *essential* when $\bigcap_{i=1}^k V_i = 0$, the origin. In this case it can be shown that the fundamental region is simplicial, so that there are exactly $k = n$ defining hyperplanes (see [1] Proposition 1.36). The following portmanteau theorem, which applies to the case of an essential cone adapted from Steerneman [16], is given without proof. Following the discussion in that paper the terms “closed” in the statement of the theorem can be taken as “essential”.

Theorem 1.3. *Let G be a subgroup of O_n . The following are equivalent*

- (i) *There is a convex cone \mathcal{C} such that $m(x, y) = \langle x, y \rangle$ for all $(x, y) \in \mathcal{C}$.*
- (ii) *There is a connected fundamental region unique up to translation under G .*
- (iii) *$G \in \mathcal{G}$ is a finite reflection group with fundamental cone \mathcal{C}_G and its interior \mathcal{C}_G° is a fundamental region.*
- (iv) *There is a closed convex cone \mathcal{C} such that $y \prec_G x \Leftrightarrow m(z, y) \leq m(z, x)$ for all $z \in \mathcal{C}$.*
- (v) *There is a closed convex cone \mathcal{C} such that $y \prec_G x$ is a cone ordering: $x, y \in \mathcal{C} \Rightarrow x - y \in \mathcal{C}$.*

In what follows it will not be enough to use only essential cones because there will be cases where the cone ordering condition is relevant but the cone is not essential.

Let us consider a simple case. Suppose that $n = 2$ and we are considering the simple group $\{e, g_1\}$ where e is the identity and $g_1 : (x_1, x_2) \mapsto (-x_1, x_2)$. The fundamental cone is $\{x : x_1 \geq 0\}$, which is inessential. Despite this, we still have the natural ordering:

$$|y_1| \leq |x_1|,$$

obtained from the cone ordering: $x, y \geq 0 \Rightarrow x - y \geq 0$. We can make the same remark for $\{e, g_2\}$ where $g_2 : (x_1, x_2) \rightarrow (x_1, -x_2)$. But the product $\{e, g_1, g_2, g_1 g_2\}$ has the fundamental cone $(x_2, x_2) : x_1, x_2 \geq 0$, which is essential, and the ordering is

$$\{|y_1| \leq |x_1|\} \cap \{|y_2| \leq |x_2|\}.$$

We could overcome this slight difficulty by abandoning the convex hull definition of majorization in Definition 1.1 and adopting the cone condition, without the necessity of the cone being closed (essential). But it is pleasing to extend the definition of G -majorization. We first need to describe the essential and inessential parts of a fundamental cone.

Definition 1.4. Let G be a finite reflection group generated by hyperplanes $\{V_j\}$ with fundamental cone $\mathcal{C}_G = \bigcap_{i=1}^k V_j^+$. Then the inessential part of \mathcal{C}_G is $\mathcal{C}_{G,0} = \bigcap_{i=1}^k V_j$ and the essential is the orthogonal complement $\mathcal{C}_{G,1} = \mathcal{C}_{G,0}^\perp \cap \mathcal{C}_G$.

For example, in the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ given as an example above acting on \mathbb{R}^2 , the situation reduces to standard majorization. There are two hyperplanes V_1 (the x_2 axis) and V_2 (the x_1 axis), and V_i^+ is the positive half plane $x_i \geq 0$. The fundamental cone \mathcal{C}_G is the positive quadrant given by $x_1, x_2 \geq 0$; the inessential part is the origin $\mathcal{C}_{G,0} = \{(0,0)\}$; and the essential part is the $\mathcal{C}_{G,1} = \mathcal{C}_{G,0}^\perp \cap \mathcal{C}_G = \mathcal{C}_G \setminus \{(0,0)\}$. In the less trivial case, where $G = \{e, g_1\}$, still acting on \mathbb{R}^2 , we have just one hyperplane, V_1 , and the fundamental cone \mathcal{C}_G is given by $x_1 \geq 0$ as described above. The inessential part is the intersection of the hyperplanes (there is only one), namely V_1 , and the essential part is $\mathcal{C}_{G,0}^\perp \cap \mathcal{C}_G = V_1^\perp \cap V_1^+ = \{(x_1, 0), x_1 \geq 0\}$ (note, V_1^\perp is the x_1 axis).

If we restrict G and vectors x, y to $\mathcal{C}_{G,0}^\perp$, then all the conditions of Theorem 1.3 apply. The extension of G -majorization is based on this.

Definition 1.5. Let G be a finite reflection group. We define essential G -majorization by $x \prec_G^+ y$ if and only if $y^+ \in \text{conv}(\{g(x^+) : g \in G\})$, where x^+, y^+ are the respective projections of x, y into $\mathcal{C}_{G,0}^\perp$.

Note that it is not necessary to redefine G , because $\text{conv}(\{g(x^+) : g \in G\}) \subseteq \mathcal{C}_{G,0}^\perp$, in any case.

It is possible to state the more general version of Theorem 1.3, dropping the requirement that the fundamental cone be closed and replacing $y \prec_G x$ with $y \prec_G^+ x$. In what follows we make the somewhat cavalier assertion that when we use $y \prec_G x$ we have the usual definition of majorization in the essential or $y \prec_G^+ x$ in the inessential case.

We are now in a position to recapture matrix descriptions of G -majorization stated simply in terms of inequalities. These come from the cone ordering version. Let $\{a_j\}$ be the vectors orthogonally defining the hyperplanes $\{V_j\}$ and let $A = \{a_{ij}\}$ be the matrix whose rows are the a_j^T . Then the cone ordering statement that $x - y \in \mathcal{C}_G$ can be written:

$$Ay \leq Ax,$$

where “ \leq ” means componentwise. Each pair x, y in \mathcal{X} has “representatives” in \mathcal{C} that we write $\tilde{x} = g_1(x), \tilde{y} = g_2(y) \in \mathcal{C}$, for some $g_1, g_2 \in G$, and we have $y \prec_G x \Leftrightarrow A\tilde{y} \leq A\tilde{x}$. In many situations we can describe the representative by simple operations. For example, when G is the symmetric group S_n operating on \mathbb{R}^n , the fundamental cone can be taken as the region given by

$$x_1 \geq x_2 \geq \cdots \geq x_n,$$

which is not essential. We map any vector $x = (x_1, \dots, x_n)^T$ to the reordered values (order statistics) $\tilde{x} = (x_{(1)}, \dots, x_{(n)})$ where

$$\tilde{x} = (x_{(1)}, \dots, x_{(n)})^T$$

and $x_{(1)} \geq \dots \geq x_{(n)}$. Then

$$A = \begin{pmatrix} 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & & \dots & 1 & -1 \end{pmatrix}.$$

In this case we have an inessential version of majorization given by:

$$\begin{aligned} y_{(1)} - y_{(2)} &\leq y_{(1)} - y_{(2)} \\ y_{(2)} - y_{(3)} &\leq y_{(2)} - y_{(3)} \\ &\vdots \\ y_{(n-1)} - y_{(n)} &\leq y_{(n-1)} - y_{(n)}. \end{aligned}$$

The standard version of majorization is arrived at by projecting onto the plane

$$\mathcal{C}_{G,0}^\perp = \left\{ x : \sum_{i=1}^n x_i = 0 \right\}.$$

When $n = 3$, the fundamental cone, in a suitable coordinate system, is an essential two dimensional cone with angle $\frac{2\pi}{6}$.

2. EXAMPLE: THE HYPEROCTAHEDRAL GROUP

The Coxeter group of type \mathcal{B}_n , also referred to as the hyperoctahedral group, is the group of signed permutations of n letters. It can be represented by $n \times n$ signed permutation matrices, and is isomorphic to the semidirect product $\mathbb{Z}_2^n \rtimes S_n$, where S_n is the symmetric (permutation) group on n entries and \mathbb{Z}_2^n can be interpreted as changing the sign of entries. The group presentation can be represented by the Dynkin diagram in Figure 1. The Dynkin diagram shows the genera-

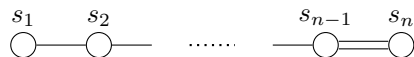


FIGURE 1. Dynkin diagram for the Coxeter group of type \mathcal{B}_n .

tors $\{s_1, \dots, s_n\}$, and relations $(s_i s_j)^{m_{ij}} = e$, where $m_{ij} = 3$ if there is a single edge between s_i and s_j and $m_{ij} = 4$ if there is a double edge. The representation of this group as signed permutations has s_i given by the 2-cycle $(i \ i+1)$ for $i = 1, \dots, n-1$ and s_n changing the sign of the n 'th coordinate. The last generator s_n is often denoted t in

the literature on Coxeter groups (sometimes being the sign change on the first coordinate). For more such information about finite reflection groups, see, for example, Humphreys [11] or Kane [12].

We now work through the case $n = 3$. The extension of the associated orders to \mathcal{B}_n is routine and also given in Section 4.3 below. The Coxeter group G of type \mathcal{B}_3 has Dynkin diagram as shown in Figure 2.

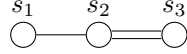


FIGURE 2. Dynkin diagram for the Coxeter group of type \mathcal{B}_3 .

Its generators $\{s_1, s_2, s_3\}$ can be represented respectively by the following signed permutation matrices operating on \mathbb{R}^3 :

$$M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The fundamental cone consistent with the ordering in the Dynkin diagram is

$$\mathcal{C}_G = \{x = (x_1, x_2, x_3)^T : x_1 \geq x_2 \geq x_3 \geq 0\}.$$

The 2-dimensional supporting hyperplanes of \mathcal{C}_G are given by

$$x_1 - x_2 = 0, \quad x_2 - x_3 = 0, \quad x_3 = 0,$$

and the fundamental cone by the region

$$x_1 - x_2 \geq 0, \quad x_2 - x_3 \geq 0, \quad x_3 \geq 0.$$

The representative \tilde{x} of some $x \in \mathbb{R}^3$ in this cone is then found by arranging the coordinates in weakly decreasing order according to their absolute values. We will denote the re-ordered coordinates

$$\tilde{x} = (x_{[1]}, x_{[2]}, x_{[3]}),$$

so that $x_{[1]}$ is the coordinate with the largest absolute value, $x_{[2]}$ is the coordinate with the next largest absolute value and so on. In other words, $|x_{[1]}| \geq |x_{[2]}| \geq |x_{[3]}| \geq 0$.

We now have an induced order $y \prec x$ given by

$$\begin{aligned} |y_{[1]}| - |y_{[2]}| &\leq |x_{[1]}| - |x_{[2]}|, \\ |y_{[2]}| - |y_{[3]}| &\leq |x_{[2]}| - |x_{[3]}|, \\ |y_{[3]}| &\leq |x_{[3]}|. \end{aligned}$$

Here,

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. SUBGROUP AND GROUP EXTENSION CONSTRUCTIONS

If N is a normal subgroup of G and H is a subgroup of G isomorphic to G/N we say that G is an extension of N by H . For general $N \triangleleft G$ it is not always the case that the quotient G/N is isomorphic to a subgroup of G (for example the quaternion group, its normal subgroup $\{\pm 1\}$ and quotient $\mathbb{Z}_2 \times \mathbb{Z}_2$), so the case of a group extension provides a special infrastructure for majorization.

We begin by describing how G -majorization can be restricted to a majorization by a subgroup H of G .

Let $\{\mathcal{X}, G\}$ define a G -majorization and let H be a subgroup of G (not necessarily normal). We define $y \prec_H x$ formally as

$$y \in \text{conv}(\mathcal{O}_H(x)).$$

We have

$$y \prec_H x \Rightarrow y \prec_G x,$$

because $H \leq G \Rightarrow \text{conv}(\mathcal{O}_H(x)) \subseteq \text{conv}(\mathcal{O}_G(x))$. We can say that \prec_G is a *refinement* of \prec_H . We can give an instructive proof using the equivalent condition from Theorem 1.2. Thus,

$$\sup_{g \in G} \langle z, g(y) \rangle = \sup_{g \in G} \langle z, gg'(y) \rangle,$$

for any fixed $g' \in G$. And similarly for x

$$\sup_{g \in G} \langle z, g(x) \rangle = \sup_{g \in G} \langle z, gg''(x) \rangle,$$

for any fixed $g'' \in G$. Now let the right cosets of H be Hg_1, Hg_2, \dots . Then

$$\sup_{g \in G} \langle z, gg'(y) \rangle = \sup_i \sup_{h \in H} \langle z, hg_i g'(y) \rangle,$$

and suppose the \sup_i is achieved at $i = r$. Then take $g' = g_r^{-1}$, and the last expression reduces to $\sup_{h \in H} \langle z, h(y) \rangle$. Carrying out a similar procedure with x and appealing to $y \prec_H x$ gives the result.

If G is an extension of N by H then we can apply this same construction to produce a majorization by the quotient G/N . In this case H is isomorphic to G/N , but the majorization depends on the isomorphism. A convenient way to approach this is to extend this isomorphism $H \cong G/N$ to a homomorphism $G \rightarrow G$ with kernel N . This can always be done, as the following (textbook) Lemma shows:

Lemma 3.1. *Let G be a group, $N \triangleleft G$ and $H \leq G$. If $\pi : G/N \rightarrow H$ is an isomorphism then π extends to a homomorphism $\phi : G \rightarrow G$ with kernel $\ker \phi = N$. Furthermore, $\text{im } \phi = H \cong G/N$.*

Proof. For $g \in G$ define $\phi(g) := \pi(gN)$. If $n \in N$ then $\phi(n) = \pi(N) = 1$, since π is a homomorphism (N is the identity of G/N), and so $N \subseteq \ker \phi$. Conversely if $g \in \ker \phi$ then $\phi(g) = \pi(gN) = 1$, but π is an isomorphism so this implies $gN = N$ and therefore $g \in N$,

completing the proof of the main statement. The claim that $\text{im } \phi \cong G/N$ is immediate from the first isomorphism theorem. \square

This shows that if G is an extension of $N \triangleleft G$ by $H \leq G$ then there is a homomorphism $\phi : G \rightarrow G$ such that $\ker \phi = N$ and $\text{im } \phi = H$. Different choices of the homomorphism ϕ may provide different subgroups $H = \text{im } \phi$, each isomorphic to G/N . To define a majorization with respect to G/N , we therefore need to take into account the map ϕ . In the same way that we like to consider G as a matrix group acting on \mathcal{X} , we can use a matrix representation $(G/N, \phi)$ of G/N which depends on ϕ . In this way there is a natural definition of majorization for G/N , depending on (N, ϕ) :

$$y \prec_{(G/N, \phi)} x \implies y \in \text{conv} \left(\mathcal{O}_{(G/N, \phi)} \right).$$

Just as it is natural to consider G acting on \mathcal{X} as a matrix group, we can consider G as the product group

$$G/N \times N \cong G.$$

We repeat, whereas the representation for N is simply induced by G , that for G/N , and consequently the majorization, depends on the particular ϕ chosen.

Let $\mathcal{V} = \bigcup V_i$ be the union of the set of reflecting hyperplanes V_i defined by the finite reflection group G acting on the space \mathcal{X} . Let \mathcal{C}_G° denote the fundamental region corresponding to G (the interior of the fundamental cone \mathcal{C}_G), as defined above. This is an open convex set with the property that each orbit of $x \in \mathcal{X} \setminus \mathcal{V}$ contains exactly one element gx (for some $g \in G$) in \mathcal{C}_G° . The set of translates $\{g\mathcal{C}_G^\circ \mid g \in G\}$ of the fundamental region is pairwise disjoint, and its union is $\mathcal{X} \setminus \mathcal{V}$.

In the case that G , N and G/N are reflection groups, we have a very simple relationship between their fundamental cones:

Theorem 3.2. *Suppose G is an extension of N by H , and that G , N and $H \cong G/N$ are all reflection groups. Then*

$$\mathcal{C}_G^\circ = \mathcal{C}_N^\circ \cap \mathcal{C}_H^\circ.$$

Proof. First note that \mathcal{C}_G° is entirely contained within \mathcal{C}_N° , since actions under elements of N are also actions of elements of G .

Consider the images of the fundamental region \mathcal{C}_G° under the action of elements of N . Since $N \leq G$ this action translates \mathcal{C}_G° into the $|N|$ disjoint translates of \mathcal{C}_N° . That is, for $n \in N$, $n\mathcal{C}_G^\circ \subseteq n\mathcal{C}_N^\circ$.

Now consider $g \in G \setminus N$, chosen so that $g \in gN \neq N$. The action of g on \mathcal{C}_G° must translate it to one of the $|N|$ regions $\{n\mathcal{C}_G^\circ \mid n \in N\}$, since the union of these regions is the whole of $\mathcal{X} \setminus \mathcal{V}$. Then $\{g\mathcal{C}_G^\circ \mid g \in gN\}$ is a set of translates of \mathcal{C}_G° , exactly one of which is in each of $\{n\mathcal{C}_N^\circ \mid n \in N\}$. For, suppose $gn_1\mathcal{C}_G^\circ$ and $gn_2\mathcal{C}_G^\circ$ are in the same $n\mathcal{C}_N^\circ$. Then $n_1\mathcal{C}_G^\circ$ and $n_2\mathcal{C}_G^\circ$ are in the same $n\mathcal{C}_N^\circ$ and therefore $n_1 = n_2$.

That is, for each coset gN and each N -translate $n\mathcal{C}_N$ there is a unique representative $g' \in gN$ with the property that $g'\mathcal{C}_G^\circ \subseteq n\mathcal{C}_N^\circ$.

Consider now the fundamental cone \mathcal{C}_H . We claim that this is equal to the union of N -translates of \mathcal{C}_G , that is

$$\mathcal{C}_H = \bigcup_{n \in N} n\mathcal{C}_G.$$

This follows because every element of G can be written uniquely as a product of an element of H with an element of N , so that

$$\bigcup_{h \in H} h \bigcup_{n \in N} n\mathcal{C}_G = \bigcup_{g \in G} g\mathcal{C}_G = \mathcal{X},$$

and because

$$h\mathcal{C}_G^\circ \cap \mathcal{C}_G^\circ = \emptyset$$

for any non-identity $h \in H$.

As a consequence, we have that

$$\mathcal{C}_N^\circ \cap \mathcal{C}_H^\circ = \mathcal{C}_N^\circ \cap \left(\bigcup_{n \in N} n\mathcal{C}_G \right)^\circ.$$

But as noted above, there is a unique N -translate of \mathcal{C}_G° inside \mathcal{C}_N° , namely \mathcal{C}_G° itself, and for all other $n \neq e$ in N we have $n\mathcal{C}_G \cap \mathcal{C}_N^\circ = \emptyset$. Therefore

$$\mathcal{C}_N^\circ \cap \left(\bigcup_{n \in N} n\mathcal{C}_G \right)^\circ = \mathcal{C}_N^\circ \cap \mathcal{C}_G^\circ = \mathcal{C}_G^\circ$$

since $\mathcal{C}_G^\circ \subseteq \mathcal{C}_N^\circ$, as required. \square

The extensive study by Maxwell [15] shows that all normal subgroups of a finite reflection group are either of index 2 in the group, or are also finite reflection groups, so that the conditions of the Theorem 3.2 are very often satisfied. Notable exceptions include the alternating subgroup A_n as a normal subgroup of the symmetric group S_n : the alternating group is not a reflection group (but it is of index 2 in S_n).

4. NORMAL SUBGROUPS IN THE HYPEROCTAHEDRAL GROUP

The normal subgroups of the group G of type \mathcal{B}_n (and other finite and affine reflection groups) are described in Maxwell [15]. For instance, the subgroup of type \mathcal{A}_n (the symmetric group S_{n+1}) and the subgroup \mathbb{Z}_2^n are both normal in G , and have quotients $G/N \cong \mathbb{Z}_2$ and S_3 respectively.

In the case $n = 3$, one composition series of G is as follows:

$$G \xrightarrow{\mathbb{Z}_2} S_4 \xrightarrow{\mathbb{Z}_2} A_4 \xrightarrow{\mathbb{Z}_3} \mathbb{Z}_2^2 \xrightarrow{\mathbb{Z}_2} \mathbb{Z}_2 \xrightarrow{\mathbb{Z}_2} 1.$$

Here S_4 is the symmetric group on 4 letters and A_4 is the alternating group on 4 letters (the group of even permutations). The labels on the arrows indicate the composition factors, so that for instance $S_4 \triangleleft G$ and $G/S_4 \cong \mathbb{Z}_2$. The composition factors of a group are unique up to isomorphism and order in the series, by the Jordan-Hölder Theorem. However there are normal subgroups that do not have simple factors and so are not featured in the composition series. For instance, $\mathbb{Z}_2^3 \triangleleft G$ and $G/\mathbb{Z}_2^3 \cong S_3$.

In this section we develop a detailed example for the case $n = 3$ in relation to these two normal subgroups (S_4 and \mathbb{Z}_2^3), including deriving the partial orders resulting from the G -majorization described above.

4.1. The normal subgroup of type \mathcal{A}_3 . The normal subgroup N of type \mathcal{A}_3 (the symmetric group S_4) is generated by the elements $\{s_1, s_2, s_3 s_2 s_3\}$ and has Dynkin diagram as shown in Figure 3.

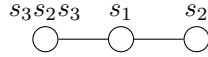


FIGURE 3. Dynkin diagram for the normal subgroup of type \mathcal{A}_3 in the Coxeter group of type \mathcal{B}_3 .

The representations are M_1 and M_2 , as for G given in Section 2 but with M_3 replaced by

$$M'_3 = M_3 M_2 M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The relations between M_1 , M_2 and M'_3 indicated by the Dynkin diagram are easily checked. The 2-dimensional supporting hyperplanes of \mathcal{C}_N are given by

$$x_1 - x_2 = 0, \quad x_2 - x_3 = 0, \quad x_2 + x_3 = 0,$$

and the fundamental cone is

$$x_1 - x_2 \geq 0, \quad x_2 - x_3 \geq 0, \quad x_2 + x_3 \geq 0.$$

Note that these inequalities are the same as the group inequalities except for the last. They imply $x_1 \geq x_2 \geq x_3$ and $x_2 \geq 0$, so the representative of $x \in \mathcal{X}$ in the cone is $\tilde{x} = (|x_{[1]}|, |x_{[2]}|, x_{[3]})$ with

$$|x_{[1]}| \geq |x_{[2]}| \geq x_{[3]}.$$

This induces an order $y \prec_N x$ as follows:

$$\begin{aligned} |y_{[1]}| - |y_{[2]}| &\leq |x_{[1]}| - |x_{[2]}|, \\ |y_{[2]}| - y_{[3]} &\leq |x_{[2]}| - x_{[3]}. \end{aligned}$$

The subgroup G/N is $N \cong \mathbb{Z}_2$ and following the discussion in Section 3 we are free, up to isomorphism, to select ϕ consistent with this quotient operation. There are various options. We can make it *dependent* on the selection of generators for G or N . For example, we could take the reflection in $x_3 = 0$ as the non-identity group element of G/N . For this the additional order is

$$|y_3| \leq |x_3|.$$

But this choice seems somewhat arbitrary, we could have used x_1 or x_2 , but in any such cases there would also be a preferred “direction”. We prefer the interesting case where the reflection is through $\{x : \sum_i x_i = 0\}$ which would lead to

$$\left| \sum_{i=1}^3 y_i \right| \leq \left| \sum_{i=1}^3 x_i \right|.$$

4.2. The normal subgroup \mathbb{Z}_2^3 . The normal subgroup $N \cong \mathbb{Z}_2^3$ can be generated by the elements $\{s_1 s_2 s_3 s_2 s_1, s_2 s_3 s_2, s_3\}$. These are sign changes in the first, second and third coordinates respectively. They naturally commute with each other, and so correspond to the rather uninteresting disconnected Dynkin diagram shown in Figure 4.

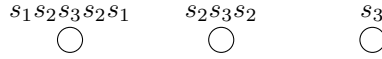


FIGURE 4. Dynkin diagram for the normal subgroup \mathbb{Z}_2^3 of the group of type \mathcal{B}_3 .

This abelian subgroup is the kernel of the map ϕ that sends $s_1 \mapsto s_1$, $s_2 \mapsto s_2$ and $s_3 \mapsto 1$. Then $G/N \cong \text{im } \phi = \langle s_1, s_2 \rangle \cong S_3$. The reflecting hyperplanes for N are simply the 2-dimensional planes orthogonal to the coordinate axes, given by

$$x_1 = 0, \quad x_2 = 0 \quad \text{and} \quad x_3 = 0.$$

The fundamental cone is then the positive eighth of \mathbb{R}^3 given by $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$. For any point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ its representative in the cone is simply $\tilde{x} = (|x_1|, |x_2|, |x_3|)$, and for any other $y \in \mathbb{R}^3$ we have the corresponding order $y \prec_N x$ given by the inequalities $|y_1| \leq |x_1|$, $|y_2| \leq |x_2|$, and $|y_3| \leq |x_3|$.

The reflecting hyperplanes for the quotient G/N are given by $x_1 - x_2 = 0$ and $x_2 - x_3 = 0$, so that this fundamental cone is $x_1 \geq x_2 \geq x_3$. Note that this cone is not essential and in particular is not contained in any of the octants of the space defined by the coordinate axes. Let $x = (x_{(1)}, x_{(2)}, x_{(3)})$ be given by ordering the coordinates so that $x_{(1)} \geq$

$x_{(2)} \geq x_{(3)}$. The inequalities defining $y \prec_{G/N} x$ are then

$$\begin{aligned} y_{(1)} - y_{(2)} &\leq x_{(1)} - x_{(2)}, & \text{and} \\ y_{(2)} - y_{(3)} &\leq x_{(2)} - x_{(3)}. \end{aligned}$$

4.3. Inequalities for the group of type \mathcal{B}_n (and \mathcal{D}_n). The inequalities in both the previous subsections extend in a straightforward manner to the general case when G is of type \mathcal{B}_n . The ordering for the group G is given by $y \prec_G x$ if and only if

$$\begin{aligned} |y_{[1]}| - |y_{[2]}| &\leq |x_{[1]}| - |x_{[2]}|, \\ &\vdots \\ |y_{[n-1]}| - |y_{[n]}| &\leq |x_{[n-1]}| - |x_{[n]}|, \\ |y_{[n]}| &\leq |x_{[n]}|. \end{aligned}$$

The $n = 3$ example of a subgroup of type \mathcal{A}_3 in the group of type \mathcal{B}_3 does not generalize to a subgroup of type \mathcal{A}_n but rather to one of type \mathcal{D}_n . This group has Dynkin diagram as shown in Figure 5.

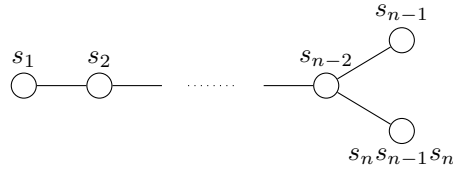


FIGURE 5. Dynkin diagram of the type \mathcal{D}_n Coxeter group, with generators showing its embedding as a normal subgroup of the group of type \mathcal{B}_n .

When $n = 3$ this diagram reduces to the three nodes on the right hand side, and hence the isomorphism with the group of type \mathcal{A}_3 in that small case (see Figure 3). The ordering derived from the normal subgroup in the general type \mathcal{B} case then gives us a set of “type \mathcal{D} ” inequalities following from those we have already obtained. For $N \triangleleft G$ of type \mathcal{D}_n the quotient is $G/N \cong \mathbb{Z}_2$, and the order $y \prec_N x$ is given by

$$\begin{aligned} |y_{[1]}| - |y_{[2]}| &\leq |x_{[1]}| - |x_{[2]}|, \\ &\vdots \\ |y_{[n-2]}| - |y_{[n-1]}| &\leq |x_{[n-2]}| - |x_{[n-1]}|, \\ |y_{[n-1]}| - |y_{[n]}| &\leq |x_{[n-1]}| - |x_{[n]}|, \end{aligned}$$

For any n the subgroup G/N is \mathbb{Z}_2 and which for the appropriate choice of generator gives our preferred version of the inequality:

$$\left| \sum_{i=1}^n y_i \right| \leq \left| \sum_{i=1}^n x_i \right|$$

Finally, when $N = \mathbb{Z}_2^n \triangleleft G$, the order $y \prec_N x$ is given by the inequalities

$$|y_1| \leq |x_1|, \quad |y_2| \leq |x_2|, \quad \dots, \quad |y_n| \leq |x_n|$$

and G/N is the symmetric group S_3 applied to X giving the inessential version of majorization.

5. G -INVARIANT FUNCTIONS

A major motivation for the study of majorization is to state inequalities for functions of interest in different fields. Formally, this means considering order-preserving functions.

Definition 5.1. *An order preserving function associated with a G -majorization is a function f such that*

$$y \prec_G x \Rightarrow f(y) \leq f(x)$$

We label the set of all such order preserving functions \mathcal{F}_G . If H is a proper subgroup of G then $\mathcal{F}_G \subset \mathcal{F}_H$. It is also clear, since $x \prec_G gx$ and $gx \prec_G x$ that any G -order preserving function, f is G -invariant: $f(x) = f(gx)$, for all $g \in G$.

Now, as above, consider a normal subgroup N and the quotient subgroup $H \cong G/N$ (so that G is an extension of N by H). For the latter we adopt one representation given by a choice of the homomorphism ϕ . We have, applying the subgroup property twice,

$$\mathcal{F}_G \subset \mathcal{F}_N \cap \mathcal{F}_H.$$

The following results shows that inclusion can be replaced by equality. First, we need an invariance result.

Lemma 5.2. *If $f \in \mathcal{F}_N \cap \mathcal{F}_H$ then f is G -invariant.*

Proof. If $g \in G$ then g can be written $g = nh$ for some $n \in N$ and $h \in H$. Then, using the fact that \prec_N and \prec_H order preserving functions are, respectively, N - and H -invariant, we have:

$$\begin{aligned} f(gx) &= f(hnx) \\ &= f(nx) \\ &= f(x). \end{aligned}$$

□

Theorem 5.3. *For a reflection group G , a normal subgroup $N \triangleleft G$ and quotient group $H = G/N$*

$$\mathcal{F}_G = \mathcal{F}_N \cap \mathcal{F}_H.$$

Proof. One inclusion has been explained, so we need to show the reverse inclusion. Assume $f_H \in \mathcal{F}_H$. We will show that $f_H \in \mathcal{F}_G$. For this we need to show that for any $\tilde{x}, \tilde{y} \in \mathcal{C}_G$ with the property $\tilde{x} - \tilde{y} \in \mathcal{C}_G$ it follows that

$$(1) \quad f_H(\tilde{y}) \leq f_H(\tilde{x}).$$

Since $\mathcal{C}_G \subset \mathcal{C}_H$, $\tilde{x} = h_1(x_1)$ and $\tilde{y} = h_2(y_1)$, for some $h_1, h_2 \in H$ and x_1, y_1 . By assumption, $\tilde{x} - \tilde{y} \in \mathcal{C}_G$ and since $\mathcal{C}_G \subset \mathcal{C}_H$ we have $\tilde{x} - \tilde{y} \in \mathcal{C}_H$. It therefore follows from the fact that $f \in \mathcal{F}_H$ that

$$(2) \quad f_H(y_1) \leq f_H(x_1).$$

Now $f_H(x_1) = f_H(h_1^{-1}\tilde{x})$. But G -invariance established in Lemma 5.2 holds, so that $f_H(x_1) = f_H(\tilde{x})$. Using the same argument we have $f_H(y_1) = f_H(\tilde{y})$. Thus (2) implies (1). Repeating the argument with a function $f_N \in \mathcal{F}_N$, completes the proof. \square

5.1. Root systems and differential conditions. Consider the case of sign changes $G = \mathbb{Z}_2^n$ and the fundamental cone is the non-negative orthant $\mathcal{C} = \{x, x \geq 0\}$. In this case the order preserving functions are functions

$$f(x_1, \dots, x_n) = \tilde{f}(|x_1|, \dots, |x_n|),$$

where \tilde{f} is an entrywise nondecreasing function on \mathcal{C} . If \tilde{f} is differentiable in each entry we can express this by $\frac{\partial \tilde{f}}{\partial x_i} \geq 0$, $i = 1, \dots, n$. Another way of saying this is that the directional derivatives *into* \mathcal{C} are non-negative, meaning in the same direction as vectors pointing into the cone from, and perpendicular to, the defining hyperplanes (in this case the coordinate hyperplanes).

The general case uses the same principle, and the arguments were well developed in Eaton and Perlman [7]. The vector directions into (and out of) the fundamental cone define the *root system*. Thus let $V_j = \{x, \langle a_j, x \rangle = 0\}$, where a_j is a unit vector, be a defining hyperplane, and $V_j^+ = \{x, \langle a_j, x \rangle \geq 0\}$ be the half-space containing \mathcal{C} . The reflection of a point x in V_j is

$$x \rightarrow x - 2\langle a_j, x \rangle x.$$

The root can be thought of as the pair $\{a_j, -a_j\}$, and the study of possible root systems is at the core of the classification of finite reflection groups (not to mention finite dimensional Lie algebras).

In the differential case the necessary and sufficient condition to be \prec_G preserving is that the directional derivatives defined by the directions a_j are non-negative. Defining $\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$, the condition is that for any $\tilde{x} \in \mathcal{C}_G$

$$\left\langle \frac{\partial \tilde{f}}{\partial \tilde{x}}, a_j \right\rangle \geq 0, \quad j = 1, \dots, n,$$

where \tilde{f} is the restriction of f to \mathcal{C} in the sense that $f(x) = \tilde{f}(\tilde{x})$ and $\tilde{x} = gx \in \mathcal{C}$.

5.2. The case G is type \mathcal{B}_n , $N = \mathbb{Z}_2^n$. The fundamental cone, \mathcal{C}_G for G is $\mathcal{C} = \{x, x_1 \geq x_2 \geq \dots \geq x_n\}$ and representatives are the ordered absolute values

$$|x_{[1]}| \geq |x_{[2]}| \geq \dots \geq |x_{[n]}|.$$

This is the intersection of \mathcal{C}_N and $\mathcal{C}_{G/N}$ given respectively by $\{x : x_1 \geq 0, \dots, x_n \geq 0\}$ and $\{x_1 \geq x_2 \geq \dots \geq x_n\}$. The cone \mathcal{C}_G , which, as expected is simplicial, is defined by taking $n - 1$ hyperplanes from $\mathcal{C}_{G/N}$ and a single hyperplane from \mathcal{C}_N , respectively:

$$\begin{aligned} |x_{[1]}| - |x_{[2]}| &= 0, \\ |x_{[2]}| - |x_{[3]}| &= 0, \\ &\vdots \\ |x_{[n-1]}| - |x_{[n]}| &= 0, \\ |x_{[n]}| &= 0. \end{aligned}$$

Each hyperplane gives a root and a differential condition and the conditions are

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial x_{[1]}} - \frac{\partial \tilde{f}}{\partial x_{[2]}} &\geq 0 \\ \frac{\partial \tilde{f}}{\partial x_{[2]}} - \frac{\partial \tilde{f}}{\partial x_{[3]}} &\geq 0 \\ &\vdots \\ \frac{\partial \tilde{f}}{\partial x_{[n-1]}} - \frac{\partial \tilde{f}}{\partial x_{[n]}} &\geq 0 \\ \frac{\partial \tilde{f}}{\partial x_{[n]}} &\geq 0 \end{aligned}$$

or, simply

$$\frac{\partial \tilde{f}}{\partial x_{[1]}} \geq \frac{\partial \tilde{f}}{\partial x_{[2]}} \geq \dots \geq \frac{\partial \tilde{f}}{\partial x_{[n]}} \geq 0,$$

on \mathcal{C} .

5.3. Examples. We consider again G to be type \mathcal{B}_n and N is \mathbb{Z}_2^n . A natural class of order preserving functions would be to take the additive form

$$\tilde{f} = \sum_{i=1}^n f_i(|x_{[i]}|),$$

where:

$$\frac{\partial \tilde{f}_1}{\partial x_{[1]}} \geq \frac{\partial \tilde{f}_2}{\partial x_{[2]}} \geq \cdots \geq \frac{\partial \tilde{f}_n}{\partial x_{[n]}} \geq 0.$$

A candidate is the weight mean

$$\frac{\sum_{i=1}^n w_i |x_{[i]}|}{\sum_{i=1}^n w_i},$$

where $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$. Except for an order reversal, this covers the Winsorised, or trimmed means obtained setting $w_i = 0, 1$. The k -Winsorised mean is

$$\frac{\sum_{i=n-k}^n |x_{[i]}|}{k},$$

which keeps the k smallest absolute values and discards the large values, or outliers. However, if we only require order-preserving with respect to N then any non-negative w_i will suffice.

The quantities $\sum_{i,j=1}^n |x_i - x_j|$, which appear in the numerator of the Gini coefficient, and $\sum_{i < j}^n (x_i - x_j)^2$, which is proportional to the sample variance, are both Schur convex and are associated with the quotient G/N . But they are not order-preserving with respect to N ; taking $n = 2$ and inspecting the shape of $x_1 - x_2$ is enough to show this fact. However we can form hybrids which, under restrictions, are order preserving for the combined ordering.

To avoid having to write $|x_{[i]}|$ let us assume that all $x_1 \geq \cdots x_n \geq 0$, for example in the case of income. We can recapture the $|x_{[i]}|$ case easily. Consider, then, for $0 \leq \alpha \leq 1$,

$$f_\alpha(x) = (1 - \alpha) \left(\frac{\sum_{i,j=1}^n |x_i - x_j|}{n^2} \right) + \alpha \left(\frac{\sum_{i=1}^n x_i}{n} \right).$$

The first term in brackets is called the *Gini mean difference* and the second term is the sample mean, \bar{x} . Moreover $f_\alpha(x)$ is Schur convex (although it is often assumed that the mean is constant). To force it to be also order preserving with respect to N we need it to be componentwise increasing in x_i , for $i = 1, \dots, n$. For this to hold α must be large enough, and we will have a condition depending on x . The condition is

$$\frac{\partial f_\alpha(x)}{\partial x_i} \geq 0, \quad i = 1, \dots, n.$$

Now,

$$\frac{\partial f_\alpha(x)}{\partial x_i} = \frac{2(1 - \alpha)}{n^2} q(x_i) + \frac{\alpha}{n}$$

where

$$q(x_i) = - \sum_{j=1, j < i}^n (x_j - x_i) + \sum_{j=1, j > i}^n (x_i - x_j),$$

The minimum is achieved when $i = n$, so substituting this we have the condition

$$\alpha \geq \frac{2q}{2q+1},$$

where

$$\begin{aligned} q &= -\frac{q(x_n)}{n} \\ &= \frac{\sum_{j=1}^{n-1} (x_j - x_n)}{n} \\ &= \bar{x} - x_n. \end{aligned}$$

REFERENCES

- [1] P. Abramenko and K.S. Brown. *Buildings: theory and applications*. Springer, 2008.
- [2] M.L. Eaton. A review of selected topics in multivariate probability inequalities. *The Annals of Statistics*, pages 11–43, 1982.
- [3] M.L. Eaton. On group induced orderings, monotone functions, and convolution theorems. In *Inequalities in statistics and probability: proceedings of the Symposium on Inequalities in Statistics and Probability, October 27-30, 1982, Lincoln, Nebraska*, volume 5, page 13. Michigan State Univ, 1984.
- [4] M.L. Eaton. *Group induced orderings with some applications in statistics*. University of Minnesota, School of Statistics, 1987.
- [5] M.L. Eaton. *Lectures on topics in probability inequalities*. Number 35. Centrum voor Wiskunde en Informatica, 1987.
- [6] M.L. Eaton. Concentration inequalities for Gauss-Markov estimators. *Journal of Multivariate Analysis*, 25(1):119–138, 1988.
- [7] M.L. Eaton and M.D. Perlman. Reflection groups, generalized Schur functions, and the geometry of majorization. *The Annals of Probability*, 5(6):829–860, 1977.
- [8] M.L. Eaton and M.D. Perlman. Concentration inequalities for multivariate distributions: I. multivariate normal distributions. *Statistics & probability letters*, 12(6):487–504, 1991.
- [9] A. Giovagnoli and HP Wynn. G -majorization with applications to matrix orderings. *Linear Algebra and its Applications*, 67:111–135, 1985.
- [10] G.H. Hardy, J.E. Littlewood, and G. Polya. *Inequalities*. Cambridge University Press, 1988.
- [11] James E. Humphreys. *Reflection groups and Coxeter groups*. Cambridge University Press, Cambridge, 1990.
- [12] R. Kane. *Reflection groups and invariant theory*. Springer, 2001.
- [13] A.W. Marshall and I. Olkin. *Inequalities: theory of majorization and its applications*, volume 143. Academic Pr, 1979.
- [14] A.W. Marshall, I. Olkin, and B.C. Arnold. *Inequalities: theory of majorization and its applications*. Springer, 2010.
- [15] G. Maxwell. The normal subgroups of finite and affine Coxeter groups. *Proceedings of the London Mathematical Society*, 76(2):359–382, 1998.
- [16] A.G.M Steerneman. G -majorization, group-induced cone orderings, and reflection groups. *Linear Algebra and its Applications*, 127:107–119, 1990.

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